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Infinity Minus Infinity

James East

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In this note, I consider an argument advanced by William Lane Craig and James D. Sinclair against the possibility of actual infinite collections based on Hilbert's Hotel and alleged problems with inverse operations in transfinite arithmetic. I aim to show that this argument is misguided, since it is based on a mistaken view that the impossibility of defining $\aleph_0 - \aleph_0$ entails the impossibility of removing an infinite subcollection from an infinite collection.

1. Introduction

Kalām Cosmological Arguments seek to establish the existence of a First Cause based on the premise that the universe had a beginning. This premise is typically supported either by empirical data (such as Big Bang cosmology) or by an argument against the possibility of an infinite past. In the *Blackwell Companion to Natural Theology* (2009), William Lane Craig and James D. Sinclair argue against the possibility of an infinite past as follows:

P1. An actual infinite cannot exist.

P2. An infinite temporal regress of events is an actual infinite.

C. Therefore, an infinite temporal regress of events cannot exist.[1]

They offer several supporting arguments for both of the premises, and this note concerns the supporting argument for P1 based on Hilbert's Hotel and an alleged problem with inverse operations in transfinite arithmetic. According to Craig, the "strongest arguments in favour of the impossibility of the existence of an actual infinite" are "those based on inverse operations performed with transfinite numbers."[2] Craig continues to favour this style of argumentation in his public lectures and debates, many of which are readily available online.

The familiar story of Hilbert's Hotel involves a hotel with infinitely many rooms, numbered 1,2,3, . . . , each of which is occupied by a guest.

[1]William Lane Craig and James D. Sinclair, "The *Kalam* Cosmological Argument," in *The Blackwell Companion to Natural Theology*, ed. William Lane Craig and J. P. Moreland (West Sussex, UK: Blackwell Publishing, John Wiley & Sons, 2009), 103.

[2]William Lane Craig, "Reply to Smith: On the Finitude of the Past," *International Philosophical Quarterly* 33 (1993).

When a new guest arrives, the proprietor asks the guest in Room N to move to Room $N+1$ (for $N = 1,2,3, \dots$), thereby freeing up Room 1 for the new guest to use. The proprietor can even accommodate (countably) infinitely many new guests by asking the guest in Room N to move to Room $2N$, thus freeing up the odd numbered rooms for the new guests. But, according to Craig and Sinclair, "Hilbert's Hotel is even stranger than the German mathematician made it out to be."[3] Indeed, as they observe, if all the guests in odd numbered rooms $(1,3,5, \dots)$ check out, there will still be infinitely many guests remaining: all those in even numbered rooms $(2,4,6, \dots)$. However, if all the guests in Rooms $4,5,6, \dots$ checked out, then the hotel would be nearly empty, with only three rooms remaining occupied.

The two scenarios above indicate that one could remove infinitely many objects from an infinite collection (if one existed) in two different ways, and end up with a different number of objects left over. And this, Craig and Sinclair allege, is "absurd. . . . Can anyone believe that such a hotel could exist in reality?"[4] Craig and Sinclair rightly note that "inverse operations of subtraction and division with infinite quantities are prohibited" in transfinite arithmetic, but protest that "in reality, one cannot stop people from checking out of a hotel if they so desire!"[5] In other words, they allege that there is some kind of disconnect between mathematics and reality, in that mathematical considerations somehow imply the impossibility of an action (checking out of a hotel) that we know should be possible (no matter how many rooms there are).

I do not claim to know that Hilbert's Hotel *could* exist in some metaphysically possible version of reality, but I aim to show that the reasons Craig and Sinclair give for rejecting the possibility are flawed. This will involve recalling the two standard methods of defining subtraction for finite quantities: as the inverse operation of addition, and via the "taking away" operation. I will explain why both methods lead to the impossibility of defining $\mu - \mu$ where μ is an infinite cardinal, but *not* to the impossibility of performing certain tasks such as checking out of an infinite hotel.

2. Subtraction as the Inverse Operation of Addition

When (finite) arithmetic is taught to young children, addition is usually the first concept taught. Students are asked to complete exercises such as $2 + 5 = \square$. As they become more advanced, they move on to exercises like $3 + \square = 9$. Since the solution to the equation $a + \square = b$ (with a and b finite) is uniquely determined by a and b, we are able to define $b - a$ as "the unique solution to the equation $a + \square = b$."

The above considerations of guests arriving at Hilbert's Hotel indicate that the equations $\aleph_0 + 1 = \aleph_0$ and $\aleph_0 + \aleph_0 = \aleph_0$ both hold. (Here, $\aleph_0$ denotes

[3] Craig and Sinclair, "The *Kalam* Cosmological Argument," 109.

[4] Ibid., 109.

[5] Ibid., 111.

the cardinality of the set {1,2,3, …} of all natural numbers.) This shows that the equation $\aleph_0 + \square = \aleph_0$ does not have a unique solution; indeed, it has infinitely many solutions. For this reason, we are unable to define the difference $\aleph_0 - \aleph_0$ as "the unique solution to the equation $\aleph_0 + \square = \aleph_0$." To put it differently, knowing that $\aleph_0$ was added to an unknown quantity to give a total of $\aleph_0$ is not enough information to deduce the value of the unknown quantity. This means that the operation of adding $\aleph_0$ is not invertible.

But this does not entail that actual infinite collections are impossible. Many real world phenomena are modelled by mathematical operations that are not invertible: squaring numbers, multiplying matrices and composing functions are all examples. Since this does not give us cause to doubt the existence of such real world phenomena, neither should we reject the possibility of an actual infinite collection simply because such collections would be modelled by non-invertible mathematical operations.

But, as mentioned above, Craig and Sinclair argue that the most severe problems concern the situation encountered when guests check out of the hotel. For does this not seem to be a case in which we *are* trying to define $\aleph_0 - \aleph_0$? This leads us to consider the second way to define subtraction.

3. Subtraction as Taking Away

Recall that subtraction (of finite quantities) may also be defined without explicit reference to addition. One way to do this is to make use of the "taking away" operation. In fact, sometimes "5–3" is read as "five take away three." To help a child calculate 5–3, a teacher will often say something like: "If you had 5 apples, and I took away 3 of them, how many would you have left?" Repeated experimentation shows that it does not matter *which* three apples are removed: there will always be two left. You could also perform the experiment with bananas rather than apples, and the answer will always be the same. If you start with any collection of 5 objects, and remove any 3 of them, you will always end up with 2. Because of this, we can define 5–3 to be "the number of objects left when you take any 3 objects away from any collection of 5 objects." These considerations are just special cases of the following basic theorems from set theory.

Theorem 1. *Suppose A is a finite set. Suppose $B \subseteq A$ and $C \subseteq A$, and that $|B| = |C|$. Then $|A \setminus B| = |A \setminus C|$.*[6]

Theorem 2. *Suppose A and B are finite sets and that $|A| = |B|$. Suppose $C \subseteq A$ and $D \subseteq B$, and that $|C| = |D|$. Then $|A \setminus C| = |B \setminus D|$.*

But note that Theorems 1 and 2 are stated in terms of *finite* sets. Theorem 1 cannot be proven in the absence of the assumption that A is finite. Likewise, the finiteness of A and B is essential in proving Theorem 2. In fact,

[6]Here, $X \subseteq Y$ means that X is a subset of Y (i.e., that all members of X are members of Y), $|X|$ denotes the cardinality of X (i.e., the number of members of X), and $X \setminus Y$ denotes the set difference of X by Y (i.e., all the members of X that are not also members of Y).

the theorems become false statements if we remove the word "finite," for we also have the following theorem.

> **Theorem 3**. *Suppose A is a countably infinite set. Then there exist subsets* $B \subseteq A$ *and* $C \subseteq A$ *such that* $|B| = |C|$ *but* $|A \setminus B| \neq |A \setminus C|$.[7]

Theorem 3 shows that it is not possible to define $\aleph_0 - \aleph_0$ as "the number of objects left when you take any $\aleph_0$ objects away from any collection of $\aleph_0$ objects."[8] The number of objects left after one removes $\aleph_0$ objects from a collection of $\aleph_0$ objects will depend on *which objects* were removed.

But, again, this does not entail that actual infinite collections are impossible. And neither does it entail that one could not remove an infinite number of objects from an infinite collection, if one existed. If the proprietor of an infinite hotel told you that infinitely many guests had just checked out, this information alone would not allow you to determine how many guests remained; the number of guests remaining would depend on *which guests* checked out. And this is very different from saying that the guests *could not* have checked out.

4. Where Is the Contradiction?

It is possible that Craig and Sinclair anticipated something like the case I made in the previous section. In a footnote, they say:

> It will not do, in order to avoid the contradiction, to assert that there is nothing in transfinite arithmetic that forbids using set difference to form sets. Indeed, the thought experiment assumes that we can do such a thing. Removing all the guests in the odd-numbered rooms always leaves an infinite number of guests remaining, and removing all the guests in rooms numbered greater than [three] always leaves three guests remaining. That does not change the fact that in such cases identical quantities minus identical quantities yields nonidentical quantities, a contradiction.[9]

Elsewhere, they say that

> the contradiction lies in the fact that one can subtract equal quantities from equal quantities and arrive at different answers.[10]

So it seems that Craig and Sinclair think that the scenario illustrated in the story of Hilbert's Hotel contradicts a principle like:

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[7]Theorem 3 may be proved for an arbitrary countably infinite set $A = \{a_1, a_2, a_3, \dots\}$ by taking $B = A$ and $C = \{a_2, a_3, a_4, \dots\}$, and noting that $|A| = |B| = |C|$, while $|A \setminus B| = 0 \neq 1 = |A \setminus C|$. The above proof works for any *Dedekind infinite* set (a set is Dedekind infinite if it is in one-one correspondence with a proper subset or, equivalently, if it contains a countably infinite subset). Any countably infinite set is Dedekind infinite. The Axiom of Countable Choice implies that any infinite set is Dedekind infinite.

[8]More generally, if μ is any infinite cardinal, then $\mu - \mu$ cannot be defined, although $\mu - \nu$ can be defined if ν is any (finite or infinite) cardinal satisfying $\nu < \mu$; in this case, $\mu - \nu = \mu$.

[9]Craig and Sinclair, "The *Kalam* Cosmological Argument," 112n12.

[10]Ibid., 112.

(i) Removing identical quantities from identical quantities yields identical quantities.

But what reason do we have to accept this principle? Notice the similarities to Theorem 2 which, stated in similar terms, says:

(ii) Removing identical quantities from identical *finite* quantities yields identical quantities.

We know that Principle (ii) can be proved mathematically. But in order to extend Principle (ii) to the stronger Principle (i), one would need to provide an argument in favour of the principle:

(iii) Removing identical quantities from identical *infinite* quantities yields identical quantities.

Note that acceptance of Principle (iii) does not require a commitment to the existence of actual infinite quantities; indeed, it is a vacuous statement if there are no actual infinite quantities. Since we have seen (in Theorem 3) that Principle (iii) would be false if infinite collections do exist, a proof of Principle (iii) (and, hence, of Principle (i)) would require a proof that there are no infinite collections. And, of course, if the goal is to use Principle (i) to prove that there are no infinite collections, then this would render the entire argument circular.

5. Conclusion

If actual infinite collections were to exist, then they would naturally have properties that were not shared by finite collections. For one obvious example, if one attempted to count through an actual infinite collection at a constant pace, then one would never finish (and this is also the case with potential infinite collections, such as a future eternity of discrete days). The story of Hilbert's Hotel simply highlights another such property that distinguishes actual infinite collections from finite ones: just knowing that an infinite subcollection has been removed from an infinite collection of objects does not allow one to determine how many objects remain. But this property itself does not entail that actual infinite collections are impossible.[11]

University of Western Sydney

[11]The author is grateful to the anonymous referees for their helpful suggestions.